ON CHEBYSHEV POLYNOMIALS AND TORUS KNOTS

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In this work we demonstrate that the q-numbers and their two-parameter generalization, the q,p-numbers, can be used to obtain some polynomial invariants for torus knots and links. First, we show that the q-numbers, which are closely connected with the Chebyshev polynomials, can also be related with the Alexander polynomials for the class $T(s,2)$ of torus knots, s being an odd integer, and used for finding the corresponding skein relation. Then, we develop this procedure in order to obtain, with the help of q,p-numbers, the generalized two-variable Alexander polynomials, and prove their direct connection with the HOMFLY polynomials and the skein relation of the latter.

1. Introduction

The relevance of knots and links to many physical [1–3] and biophysical [4] systems implies the importance of investigating the properties and characteristics of knot-like structures. The concepts of knot theory play important role in the models of statistical physics [5], quantum field theory [6], quantum gravity [7] and in a number of other physical phenomena. In the preprint of 1975 it was proposed by L.D. Faddeev that knot-like solitons could be realized in a nonlinear field theory [8], in a definite model defined in $3+1$ dimensions. The model includes the standard nonlinear $O(3)$ σ-model, which admits static solitons in $2+1$ dimensions, and a Skyrme term. In the Faddeev model static solitons are stabilized by the integer-valued Hopf charge. Interest in the model was renewed in 1997 after an article of Faddeev and Niemi in Nature [9]. They have made first attempts at a numerical construction of solitons with the minimal energy in the form of knots. Battye and Sutcliffe demonstrated that for higher Hopf charge twisted, knotted and linked configurations occur [10], in particular, they showed that the minimal energy soliton with Hopf charge seven is a trefoil knot.

R.J. Finkelstein has proposed a field theory model, in which local $SU(2)\times U(1)$, the symmetry group of the standard electroweak theory, is combined with global quantum group $SU_q(2)$, the symmetry group of knotted solitons [11,12]. This allows to incorporate the q-soliton into the field theory and to replace the point particles by knotted solitons. More recent discussion on the role of field theory knots both in superconductivity theory and in the Yang-Mills theory can be found in [13].

In the context of modelling static properties of hadrons, it was shown in [14] (see also [15]) that global quantum groups $SU_q(n)$, $n=2,\ldots,6$, can be successfully applied for flavor symmetries, and certain torus knots put into correspondence, through Alexander polynomials, with vector quarkonia.

Various polynomial invariants are known to be one of the basic characteristics of knots and links (see e.g. [16]). Among them, the Alexander polynomials, the Jones polynomials and the HOMFLY polynomials are both best studied and play important role in the knot theory and its applications.

To describe some properties and characteristics of knots and links the classical Chebyshev polynomials can be used. For example, in [17] the Chebyshev polynomials were utilized for polynomial parametrization of noncompact counterparts of torus knots. It was shown how to construct the Chebyshev model associated with any knot.

In this paper we concentrate on studying the polynomial invariants such as the Alexander polynomials and the HOMFLY polynomials, and their close connection with the Chebyshev polynomials. We restrict ourselves with the set of torus knots and links, and show that certain rather simple two-variable generalization of the Chebyshev polynomials is well suited for characterizing those knots.

2. Alexander polynomials and skein relation

The Alexander polynomials $A(t)$ for knots and links can be defined (see e.g. [16]) by the skein relation

$$A_+(t)=(t^{\frac{1}{2}}-t^{-\frac{1}{2}})A_O(t)+A_-(t)\,, \qquad (1)$$

and the condition for the unknot:

$$A_{unknot}=1\,. \qquad (2)$$

Using (1) and (2), one can find the Alexander polynomial for any knot or link applying to it, in a standard way, the surgery operations of switching and elimination.

From now on we consider the torus knots and links of type $(s,2)$, where s is any positive integer. If $s=1$ we have the unknot, the case of $s=2$ corresponds to the Hopf link, and $s=3$ to the trefoil knot, and so on. In general, when s is odd, we have the series of torus knots $T(s,2)$, and if s is even, we have the series of two-component torus links $L(s,2)$. Here s equals the minimal number of crossings.

Applying the operation of elimination to $(s,2)$, one obtains $(s-1,2)$, whereas the switching operation turns $(s,2)$ into $(s-2,2)$, for $s>2$. This means that $A_+(t)$, $A_O(t)$ and $A_-(t)$ correspond to three successive Alexander polynomials, which allows to make the following juxtaposition in (1)

$$\begin{aligned} &A_+(t)\to \widetilde{A}^2_{n+1}(t)\,, \quad A_O(t)\to \widetilde{A}^2_n(t)\,,\\ &A_-(t)\to \widetilde{A}^2_{n-1}(t)\,. \end{aligned} \tag{3}$$

Thus, from (1) and (3) one obtains the recurrence relation for the tilded Alexander polynomials for the unified set of torus knots and links of type $(s,2)$, (the polynomials are arranged by increasing degrees):

$$\widetilde{A}^2_{n+1}(t)=(t^{\frac12}-t^{-\frac12})\widetilde{A}^2_n(t)+\widetilde{A}^2_{n-1}(t)\,. \tag{4}$$

It is convenient to denote the Alexander polynomials for the subset containing only torus knots (or the subset of torus links) as

$$A^{s,2}_m(t)\equiv A^{s,2}_m\equiv A^2_m\,.$$

Here m is the degree of the corresponding Alexander polynomial, which has the form of Laurent polynomial:

$$m=\frac12(s-1)\,. \tag{5}$$

Since $s=2m+1$ for both knots and links, the degree m of the Alexander polynomial for knots $T(s,2)$ is an integer, and m for links $L(s,2)$ is half-integer. Let us first give the table of the Alexander polynomials $A^{s,2}_m(t)$ for torus knots $T(s,2)\equiv T(2m+1,2)$

$$\begin{aligned} &A^{1,2}_0(t)=1\,,\\ &A^{3,2}_1(t)=t-1+t^{-1}\,,\\ &A^{5,2}_2(t)=t^2-t+1-t^{-1}+t^{-2}\,,\\ &A^{7,2}_3(t)=t^3-t^2+t-1+t^{-1}-t^{-2}+t^{-3}\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &A^{2m+1,2}_m(t)=t^m-t^{m-1}+\cdots-t^{-(m-1)}+t^{-m}=\\ &=\sum_{i=0}^{m}t^{m-2i}-\sum_{i=0}^{m-1}t^{m-2i-1}=\sum_{i=0}^{2m}(-1)^i t^{m-i}\,. \end{aligned} \tag{6}$$

Recurrence formula for the polynomials (6) looks as (dropping $2m+1$ in superscript)

$$A^2_{m+1}(t)=(t+t^{-1})A^2_m(t)-A^2_{m-1}(t)\,. \tag{7}$$

Now consider torus links of the type $L(s,2)\equiv L(2m+1,2)$, where s is positive even integer. The degree m of the Alexander polynomial $A^{s,2}_m(q)$ is again as in (5). It is half-integer now. The table shows the Alexander polynomials for these torus links:

$$\begin{aligned} &A^{2,2}_{\frac12}(t)=t^{\frac12}-t^{-\frac12},\\ &A^{4,2}_{\frac32}(t)=t^{\frac32}-t^{\frac12}+t^{-\frac12}-t^{-\frac32}\,,\\ &A^{6,2}_{\frac52}(t)=t^{\frac52}-t^{\frac32}+t^{\frac12}-t^{-\frac12}+t^{-\frac32}-t^{-\frac52}\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &A^{2m+1,2}_m(t)=t^m-t^{m-1}+\cdots+t^{-(m-1)}-t^{-m}=\\ &=\sum_{i=0}^{2m}(-1)^i t^{m-i}\,. \end{aligned} \tag{8}$$

Note, the polynomials (8) satisfy the recurrence relation (7) too, though now with half-integer subscripts. So, for torus knots and links of type $(s,2)$, eq. (7) describes either three successive torus knots (if s is an odd integer), or three successive torus links (if s is an even integer). Unifying two tables (6) and (8), we have the table of the Alexander polynomials for torus knots $T(s,2)$ and links $L(s,2)$, where $s=2m+1$ is an integer, while m is an integer or half-integer and equals the degree of the Alexander polynomial:

$$\begin{aligned} &\widetilde{A}^{1,2}_0(t)\equiv A^{1,2}_0(t)=1\,,\\ &\widetilde{A}^{2,2}_1(t)\equiv A^{2,2}_{\frac12}(t)=t^{\frac12}-t^{-\frac12},\\ &\widetilde{A}^{3,2}_2(t)\equiv A^{3,2}_1(t)=t-1+t^{-1}\,,\\ &\widetilde{A}^{4,2}_3(t)\equiv A^{4,2}_{\frac32}(t)=t^{\frac32}-t^{\frac12}+t^{-\frac12}-t^{-\frac32}\,,\\ &\widetilde{A}^{5,2}_4(t)\equiv A^{5,2}_2(t)=t^2-t+1-t^{-1}+t^{-2}\,,\\ &\widetilde{A}^{6,2}_5(t)\equiv A^{6,2}_{\frac52}(t)=\\ &\qquad=t^{\frac52}-t^{\frac32}+t^{\frac12}-t^{-\frac12}+t^{-\frac32}-t^{-\frac52}\,,\\ &\widetilde{A}^{7,2}_6(t)\equiv A^{7,2}_3(t)=\\ &\qquad=t^3-t^2+t-1+t^{-1}-t^{-2}+t^{-3}\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\widetilde{A}^{2m+1,2}_{2m}(t)\equiv A^{2m+1,2}_m(t)=\\ &=t^m-t^{m-1}+t^{m-2}-\cdots t^{-m}=\sum_{i=0}^{2m}(-1)^i t^{m-i}\,. \end{aligned} \tag{9}$$

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Two different notations for the Alexander polynomials are related as

$$\widetilde{A}^{2m+1,2}_{2m}(t) = A^{2m+1,2}_{m}(t)\,.$$

Recurrence relation for the polynomials (9) is given by (4), which immediately follows from the skein relation (1), in view of the correspondence (3).

Let us show that (4) can be obtained from (7) as well. To see that, we first write (1) in general form

$$A_+(t) = b_1 A_O(t) + b_2 A_-(t)\,. \tag{10}$$

With the account of (3), we also have the recursion

$$\widetilde{A}^2_{n+1}(t) = b_1 \widetilde{A}^2_n(t) + b_2 \widetilde{A}^2_{n-1}(t)\,. \tag{11}$$

Then rewrite eq. (7) in terms of tilded $\widetilde{A}^2_n(t)$:

$$\widetilde{A}^2_{n+1}(t) = (t+t^{-1})\widetilde{A}^2_{n-1}(t) - \widetilde{A}^2_{n-3}(t)\,. \tag{12}$$

The latter in general terms looks as

$$\widetilde{A}^2_{n+1}(t) = c_1 \widetilde{A}^2_{n-1}(t) + c_2 \widetilde{A}^2_{n-3}(t)\,. \tag{13}$$

From (11) we have

$$\widetilde{A}^2_{n}(t) = b_1 \widetilde{A}^2_{n-1}(t) + b_2 \widetilde{A}^2_{n-2}(t)\,, \tag{14}$$

$$\widetilde{A}^2_{n-1}(t) = b_1 \widetilde{A}^2_{n-2}(t) + b_2 \widetilde{A}^2_{n-3}(t)\,. \tag{15}$$

Insert (14) into (11)

$$\widetilde{A}^2_{n+1}(t) = (b_1^2 + b_2)\widetilde{A}^2_{n-1}(t) + b_1 b_2 \widetilde{A}^2_{n-2}(t)\,. \tag{16}$$

Then put $\widetilde{A}^2_{n-2}(t)$ from (15) into (16)

$$\widetilde{A}^2_{n+1}(t) = (b_1^2 + 2b_2)\widetilde{A}^2_{n-1}(t) - b_2^2 \widetilde{A}^2_{n-3}(t)\,. \tag{17}$$

Comparison of (17) and (13) gives

$$c_1 = b_1^2 + 2b_2\,, \quad c_2 = -b_2^2\,. \tag{18}$$

From eq. (18),

$$b_1 = (c_1 - 2b_2)^{\frac{1}{2}}\,, \quad b_2 = (-c_2)^{\frac{1}{2}}\,. \tag{19}$$

Note that the latter two formulas, which involve general coefficients, will be used below (see Sec. 4). Comparing (12) and (13) yields

$$c_1 = t + t^{-1}\,, \quad c_2 = -1\,.$$

With account of (19) this implies

$$b_2 = 1\,, \quad b_1 = t^{\frac{1}{2}} - t^{-\frac{1}{2}}\,,$$

which coincides with the coefficients in (4), and thus our statement is proved.

Since the formulas (18) and (19) connect arbitrary pairs b_1, b_2 and c_1, c_2, this allows one to gain general skein relation from the corresponding recurrence relation for the set of torus knots $T(s,2)$.

3. Alexander polynomials from Chebyshev polynomials

In this section we describe the connection between Alexander polynomials and Chebyshev polynomials, using the q-numbers. The q-number corresponding to the integer n is defined as (see e.g. [18, 19] and [20])

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,, \tag{20}$$

where q is a parameter. If $q \to 1$, then $[n]_q \to n$. Considering q to be variable, we go over to the q-polynomials. Some of the q-numbers (or q-polynomials) are as follows:

$$[1]_q = 1\,, \quad [2]_q = q + q^{-1}\,,$$

$$[3]_q = q^2 + 1 + q^{-2}\,, \quad [4]_q = q^3 + q + q^{-1} + q^{-3}\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)} = \sum_{i=0}^{n-1} q^{n-1-2i}\,. \tag{21}$$

The (easily verifiable) recurrence relation for the q-numbers (or q-polynomials) is

$$[n+1]_q = (q + q^{-1})[n]_q - [n-1]_q\,. \tag{22}$$

From now on, rename the variable in the Alexander polynomials:

$$t \to q\,.$$

From comparison, for the considered class of torus knots, of the Alexander polynomials and the q-polynomials, we see that the Alexander polynomials (6) can be expressed through q-polynomials (21) in the simple way:

$$A^2_n(q) = [n+1]_q - [n]_q\,. \tag{23}$$

This relation for the Alexander polynomials was found in [14, 15], in the context of their correspondence to masses of vector quarkonia.

Below we will need some properties of the classical Chebyshev polynomials, in order to formulate the Alexander polynomials in terms of the Chebyshev ones. If $x = 2\cos\theta$, the Chebyshev polynomials of the first kind are defined as

$$T_n(x) = 2\cos(n\theta)\,. \tag{24}$$

From (24), some first cases read

$$T_0=2\,,\quad T_1=x\,,\quad T_2=x^2-2\,,\quad T_3=x^3-3x\,,\quad \ldots\,.$$

The recurrence formula is known as

$$T_{n+1}=xT_n-T_{n-1}\,. \qquad (25)$$

Chebyshev polynomials of the second kind are

$$V_n(x)=\frac{\sin((n+1)\theta)}{\sin\theta}\,. \qquad (26)$$

Polynomials T_n and V_n are both monic and have the degree n. From (26) we have

$$V_0=1\,,\quad V_1=x\,,\quad V_2=x^2-1\,,\quad V_3=x^3-2x\,,\quad \ldots\,,$$

and the recurrence relation is

$$V_{n+1}=xV_n-V_{n-1}\,. \qquad (27)$$

There is a connection between (24) and (26)

$$T_n(x)=V_n(x)-V_{n-2}(x)\,. \qquad (28)$$

Putting

$$q=e^{i\theta} \qquad (29)$$

into (20), we have

$$[n]_q=\frac{\sin(n\theta)}{\sin\theta}=V_{n-1}(x)\,, \qquad (30)$$

where $V_n(x)$ is the Chebyshev polynomial of the second kind, and

$$x=2\cos\theta=q+q^{-1}\,. \qquad (31)$$

From (30), (31) it is seen that

$$V_n(q)=[n+1]_q\,, \qquad (32)$$

and therefore (23) takes the form

$$A_n^2(q)=V_n(x)-V_{n-1}(x)\,,\qquad x=q+q^{-1}. \qquad (33)$$

Thus, the Alexander polynomials $A_n^2(q)$ are obtained from the Chebyshev polynomials of the second kind $V_n(x)$ (26), after changing the variables $x\to q+q^{-1}$, by means of the formula (33).

4. Generalized Alexander polynomials and HOMFLY polynomials

Now let us put into consideration the q,p-numbers, a natural generalization of q-numbers. With the help of q,p-numbers we will construct a generalization of the Alexander polynomials – $A_n^2(q,p)$, which now depend on two variables q,p. Afterwards we intend to show that $A_n^2(q,p)$ turn into the well-known HOMFLY polynomials by an appropriate change of variables.

The q,p-number corresponding to the integer number n is defined as (see e.g. [21])

$$[n]_{q,p}=\frac{q^n-p^n}{q-p}\,, \qquad (34)$$

where $q\,,p$ are some complex parameters. If $p=q^{-1}$, then $[n]_{q,p}=[n]_q$. Some of the q,p-numbers are

$$[1]_{q,p}=1\,,\quad [2]_{q,p}=q+p\,,$$

$$[3]_{q,p}=q^2+qp+p^2\,,\quad [4]_{q,p}=q^3+q^2p+qp^2+p^3\,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[n]_{q,p}=q^{n-1}+q^{n-2}p+q^{n-3}p^2+\cdots+$$
$$+qp^{n-2}+p^{n-1}=\sum_{i=0}^{n-1}q^{n-1-i}p^i=q^{n-1}\sum_{i=0}^{n-1}q^{-i}p^i\,. \qquad (35)$$

Considering q and p as variables, we deal with q,p-polynomials. Then, the recurrence relation for them is

$$[n+1]_{q,p}=(q+p)[n]_{q,p}-qp[n-1]_{q,p}\,. \qquad (36)$$

On the base of eq. (32) and the expression (34) or (35) for the q,p-polynomials , we introduce a natural generalization of the Chebyshev polynomials of the second kind, which now depend on the two variables:

$$V_n(q,p)=[n+1]_{q,p}\,. \qquad (37)$$

From (36), (37) the recurrence relation does follow:

$$V_{n+1}(q,p)=(q+p)V_n(q,p)-qpV_{n-1}(q,p)\,. \qquad (38)$$

Now, in analogy with (33), we introduce the two-variable generalized Alexander polynomial as linear combination of the polynomials (37). Due to this proposal, the following recurrence formula takes place:

$$A_{n+1}^2(q,p)=(q+p)A_n^2(q,p)-qpA_{n-1}^2(q,p)\,. \qquad (39)$$

This is a direct analog of (7) and reduces to it if $q=t$ and $p=t^{-1}$. To continue the analogy we take

$$A_0^2(q,p)=1\,,\quad A_1^2(q,p)=q-qp+p\,. \qquad (40)$$

It is easy to see that (39) with (40) will be valid if

$$A_n^2(q,p) = V_n(q,p) - qpV_{n-1}(q,p) = \\ = [n+1]_{q,p} - qp[n]_{q,p}\,. \tag{41}$$

Setting

$$q = re^{i\theta}\,, \quad p = \bar{q} = re^{-i\theta} \tag{42}$$

into (34), we have

$$[n]_{q,p} = \frac{r^n \sin(n\theta)}{r\sin\theta} = r^{n-1}V_{n-1}(x)\,. \tag{43}$$

If $r=1$, eq. (43) turns into (30). Taking into account (37) and (43), we obtain $V_n(q,p)$ with a factorized form of dependence on the variables r, x :

$$V_n(r,x) = r^n V_n(x)\,. \tag{44}$$

Here $V_n(x)$ is the classical Chebyshev polynomial of second kind, with x as in (31). The corresponding two-variable Chebyshev polynomials of first kind also arise:

$$T_n(r,x) = 2r^n\cos(n\theta)\,.$$

In the variables r, x, see (42) and (31), the recurrence relation (39) is written as

$$A_{n+1}^2(r,x) = rxA_n^2(r,x) - r^2A_{n-1}^2(r,x)\,. \tag{45}$$

The first two polynomials (40) become

$$A_0^2(r,x) = 1\,, \quad A_1^2(r,x) = rx - r^2\,. \tag{46}$$

From (41) and (44) we also have

$$A_n^2(r,x) = r^n(V_n(x) - rV_{n-1}(x))\,. \tag{47}$$

Now we make a key proposal: we apply the generalized Alexander polynomials $A_n^2(r,x)$, given by (45) and (46), for describing the torus knots $T(s,2)$. From (19), with account of (42), we have

$$c_1 = rx\,, \quad c_2 = -r^2\,,$$

and then

$$b_2 = r\,, \quad b_1 = (rx-2r)^{\frac{1}{2}} = r^{\frac{1}{2}}(x-2)^{\frac{1}{2}}\,. \tag{48}$$

Hence, as a generalization of (1), from (10), (11) and (48), we obtain the skein relation for the generalized Alexander polynomials:

$$A_+(r,x) = r^{\frac{1}{2}}(x-2)^{\frac{1}{2}}A_O(r,x) + rA_-(r,x)\,. \tag{49}$$

Now let us explore the connection between the generalized Alexander skein relation (49) and the HOMFLY skein relation. By definition, the HOMFLY polynomials $H(a,z)$ satisfy the skein relation

$$a^{-1}H_+(a,z) - a^1H_-(a,z) = zH_O(a,z)\,,$$

or, in equivalent form,

$$H_+(a,z) = azH_O(a,z) + a^2H_-(a,z)\,, \tag{50}$$

with $H_{unknot}=1$. As before, consider the torus knots $T(s,2)$, s odd integer. For these, the notation for the corresponding HOMFLY polynomials is similar to the notation for the Alexander ones, namely

$$H(s,2)(a,z) \equiv H(2m+1,2)(a,z) \equiv H_m^2(a,z) \equiv H_m^2\,.$$

The short list of the HOMFLY polynomials for torus knots $T(s,2) \equiv T(2m+1,2)$ is:

$$\begin{aligned} &H_0^{1,2}(a,z) = 1\,,\\ &H_1^{3,2}(a,z) = 2a^2 + a^2z^2 - a^4\,,\\ &H_2^{5,2}(a,z) = 3a^4 + 4a^4z^2 + a^4z^4 - 2a^6 - a^6z^2\,,\\ &H_3^{7,2}(a,z) = 4a^6 + 10a^6z^2 + 6a^6z^4 + a^6z^6 - 3a^8 - \\ &\qquad\qquad -4a^8z^2 - a^8z^4\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\,. \end{aligned} \tag{51}$$

Recurrence relation for (51) reads

$$H_{m+1}^2(a,z) = a^2(z^2+2)H_m^2(a,z) - a^4H_{m-1}^2(a,z)\,. \tag{52}$$

If we compare (52) and (45), we see that through the substitution

$$r = a^2\,, \quad x = z^2 + 2\,, \tag{53}$$

the HOMFLY polynomials and the generalized Alexander polynomials coincide:

$$H_n^2(a,z) = A_n^2(r,x) = r^n(V_n(x) - rV_{n-1}(x))\,.$$

Then, the HOMFLY skein relation (50) in the variables r, x, see (53), looks as

$$H_+(r,x) = r^{\frac{1}{2}}(x-2)^{\frac{1}{2}}H_O(r,x) + rH_-(r,x)\,, \tag{54}$$

which coincides with (49). Besides,

$$A_0^2(r,x) = H_0^2(r,x)\,, \quad A_1^2(r,x) = H_1^2(r,x)\,.$$

Thus, we have proved that the generalized Alexander polynomials and their skein relation go over into the HOMFLY ones by applying the parametrization (53). On the other hand, the HOMFLY skein relation and polynomials turn into the generalized Alexander ones with the help of inverse substitution

$$a = r^{\frac{1}{2}}\,, \quad z = (x-2)^{\frac{1}{2}}\,.$$

5. Concluding remarks

We have demonstrated that the connection of the Chebyshev polynomials with the Alexander polynomials can be realized in a rather simple way if one uses, as an auxiliary tools, the concept of q-numbers. On the other hand, existence of the q,p-numbers, which generalize the q-numbers, makes it possible to generalize the Chebyshev polynomials to their two-variable modification and, by exploiting the analogy with previous one-variable case, also to achieve two-variable generalization of the Alexander polynomials. Finally, we have found that the two-variable extended Alexander polynomials are mapped onto the HOMFLY polynomials.

We hope the proposed way of usage of the Chebyshev polynomials will be helpful for further investigation of knots and links, not only on the base of the Alexander polynomials (along with their two-variable modification) and HOMFLY polynomials treated above, but also possibly in connection with Kauffman polynomials and other known polynomial invariants. Besides, it is of interest to study, within the proposed scheme, the (s, r) torus knots more general than the particular class $(s, 2)$ considered in this paper. Subsequently we hope to use the explored polynomial invariants within the framework of some physical models.

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ПРО ПОЛІНОМИ ЧЕБИШОВА І ТОРИЧНІ ВУЗЛИ

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Р е з ю м е

В роботі показано, що q-числа та їх двопараметричні узагальнення, q, p-числа, можна використати для отримання деяких поліноміальних інваріантів торичних вузлів і зачеплень. По-перше, показується, що q-числа, які тісно пов'язані з поліномами Чебишова, можуть бути пов'язані з поліномами Александера для класу $T(s, 2)$ торичних вузлів, де s - непарне ціле число, і використані для знаходження відповідного скейн-співвідношення. Затим, ми застосовуємо цю процедуру для отримання, за допомогою q, p-чисел, двопараметричних узагальнених поліномів Александера, та показуємо зв'язок останніх із поліноміальними інваріантами HOMFLY та їх скейн-співвідношенням.